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On the Uniqueness of the Tetrahedral Association Schemes

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Let V be the set of $\binom{n}{3}$ 3- sets in $\{1 \cdots n\}$. Say $p, q \in V$ are ith associates, $(p, q) \in A_i$, if $3 = i + |p \cap q|$. An association scheme is tetrahedral if it is isomorphic to the scheme $\{A_0, A_1, A_2, A_3\}$ and a graph is tetrahedral if it is isomorphic to A_1. Aigner [1] and Bose and Laskar [2] have shown that the tetrahedral graphs are characterized by their characteristic equations, provided $n < 9$ or $n > 16$. The present paper extends methods of Hoffman [7] to show that the tetrahedral association schemes are characterized by their structural constants, provided $n > 10$.

1. INTRODUCTION

The concept of an association scheme [5] represents a significant link between combinatorics and algebra. The importance of certain association schemes to the study of permutation groups was already recognized by Schur [9] but only recently, [6, 10] has the precise relationship between these subjects been explained.

Unfortunately permutation groups seem inappropriate tools for the study of association schemes in their own right. In fact, with one important exception, only elementary counting arguments have been successfully used to attack the central problem of characterizing a known scheme or family of schemes in terms of their parameters. The exception is, of course, Hoffman's remarkable paper [7] in which certain subconfigurations of a pseudo-triangular scheme are ruled out by means of a classical theorem on the spectrum of a real symmetric matrix:

1.1. THEOREM [8, Theorem 3.2.1]. *Let A be a real symmetric matrix and suppose B is a principal submatrix of A. Suppose x is an eigenvector of B associated with the minimal eigenvalue λ of B. If μ is the minimal eigenvalue of A then*

(a) $\mu \leqslant \lambda$ *and*

(b) *if $\mu = \lambda$ then the natural injection of x from the ambient space of B into the ambient space of A is an eigenvector of A.*

The purpose of this paper is to set out an extension of Hoffman's method to schemes having many classes by proving

1.2. THEOREM. *A pseudotetrahedral association scheme on $\binom{n}{3}$ objects is tetrahedral, unless perhaps $n = 9$ or 10.*

(See Section 2 for definitions.) This theorem supplements the work of Aigner [1], and Bose and Laskar [2] on pseudotetrahedral graphs by strengthening both the hypotheses and conclusions of their theorems.

One of the difficulties encountered in applying Hoffman's method is that the spectrum of the incidence matrix of a graph has not yet been given a reasonable combinatorial interpretation. Consequently, the subconfigurations ruled out by Theorem 1.1a can only be determined by tedious calculation.

For this reason, Theorem 1.1b seems to be more helpful than Theorem 1.1a. A known association scheme may be studied to find small subconfigurations that correspond to eigenvectors of A having minimal eigenvalue, then Theorem 1.1b can be used to describe the possible embeddings of such a critical configuration in an arbitrary scheme having the same parameters as the original one. In order for this approach to work best, the matrix A must have a large eigenspace associated with its minimal eigenvalue. This becomes easier to arrange for schemes having more classes.

This approach is used in Section 3 to study the ways a certain four-object subconfiguration can be embedded in a pseudotetrahedral association scheme. If the scheme has characteristic >10 we eventually show (Theorem 3.10) that these four objects and those that are third associates of none of the original four must be a tetrahedral subscheme of characteristic 6.

Our proof of Theorem 1.2 follows that of Bose and Laskar [2] relying on large cliques of first associates, but differs in the method of construction of such cliques.

Thanks are due my colleagues, J. W. Freeman and R. C. Bose, from whose careful scrutiny and continued interest this work has greatly benefited. I also wish to thank A. J. Hoffman for pointing out the relationship between this paper and unpublished work of P. Rolland, to whom the cases $n = 15, 16$ of Theorem 1.2 should apparently be credited.

2. PRELIMINARIES

It is essential that we view association schemes in two different ways. The equivalence of our combinatorial and algebraic points of view is explained by Bose and Mesner [4]. Our primary purpose is to fix notation.

A combinatorist might describe an m-class *association scheme A* on v objects V as a system of $m + 1$ symmetric graphs $\{A_0, ..., A_m\}$ all with vertex set V such that:

(0) A_0 is the identity graph (with each vertex joined to itself only),

(1) any pair of vertices is joined in precisely one of the graphs A_i, and

(2) if a and b are ith associates (joined in the ith graph A_i), then the number of objects that are jth associates of a and kth associates of b is independent of the pair (a, b) of ith associates.

The constant postulated in (2) is denoted p^i_{jk} and the set of all p^i_{jk}, $i, j, k = 0,\dots, m$ is called the set of *parameters* of the association scheme A.

An algebraist would probably confuse the graph A_i with its incidence matrix $\mathscr{B}_i$ and view the above scheme as a $\mathbb{Z}$-algebra $\mathscr{A}$ of symmetric $v \times v$ matrices having a distinguished $\mathbb{Z}$-module basis $\{\mathscr{B}_i\}$ of 0–1 matrices such that: (0′) $\mathscr{B}_0 = I$ — the identity matrix, and (1′) $\sum \mathscr{B}_i = J$ — the matrix of all 1's. From this point of view the scheme's parameters are simply the structure constants of $\mathscr{A}$, and the objects V are the natural basis vectors of the ambient space $W \simeq \mathbb{Q}^v$ of $\mathscr{A}$.

Two association schemes A, A' are isomorphic if there is a function $f\colon V \to V'$ that is an isomorphism of A_i with A_i' for each i.

An association scheme is called *tetrahedral* of *characteristic* n if it is isomorphic with the scheme $A^* = \{A_0^*, A_1^*, A_2^*, A_3^*\}$ having the $\binom{n}{3}$ 3 sets in an n set as objects and having $(p, q) \in A_i^*$ whenever $|p \cap q| + i = 3$. A scheme A is *pseudotetrahedral* if it has the same parameters as A^*. It is an easy exercise to verify:

2.1. Proposition. *Suppose A is a pseudotetrahedral of characteristic $n \geq 6$, having associated algebra $\mathscr{A}$ generated by $\{\mathscr{B}_0 = I, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}$. Then*

$$\mathscr{B}_i\mathscr{B}_j = \sum p^k_{ij}\mathscr{B}_k,$$

where

$$P_0 = (p^k_{0j})_{j,k} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_1 = (p^k_{1j})_{j,k} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3(n-3) & (n-2) & 4 & 0 \\ 0 & 2(n-4) & 2(n-4) & 9 \\ 0 & 0 & (n-5) & 3(n-6) \end{pmatrix},$$

$$P_2 = (p^k_{2j})_{j,k} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2(n-4) & 2(n-4) & 9 \\ 3\binom{n-3}{2} & (n-4)^2 & \dfrac{(n-5)(n+2)}{2} & 9(n-6) \\ 0 & \binom{n-4}{2} & (n-5)(n-6) & 3\binom{n-6}{2} \end{pmatrix},$$

and

$$P_3 = (p_{3j}^k)_{j,k} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & (n-5) & 3(n-6) \\ 0 & \binom{n-4}{2} & (n-5)(n-6) & 3\binom{n-6}{2} \\ \binom{n-3}{3} & \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} \end{pmatrix}$$

In order to avoid repetition we fix the above notation. Thus, A is to be a pseudotetrahedral association scheme of characteristic $n \geqslant 6$, etc. The tetrahedral association scheme of characteristic n will be denoted A^*, the set of 3 sets in $\{1,..., n\}$ will be denoted V^*, etc.

2.2. Proposition. *Suppose $a, b, c, d \in \mathbb{Z}$. Then the eigenvalues of $\mathscr{B} = a\mathscr{B}_0 + b\mathscr{B}_1 + c\mathscr{B}_2 + d\mathscr{B}_3 \in \mathscr{A}$ are the coordinates of $(a, b, c, d)\mathscr{E}$ where*

$$\mathscr{E} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3(n-3) & (2n-9) & (n-7) & -3 \\ 3\binom{n-3}{2} & \dfrac{(n-4)(n-9)}{2} & -(2n-11) & 3 \\ \binom{n-3}{3} & -\binom{n-4}{2} & (n-5) & -1 \end{pmatrix}.$$

The associated eigenspaces have dimensions 1, $n-1$, $\binom{n}{2} - n$, *and* $\binom{n}{3} - \binom{n}{2}$. *In particular,* $\mathscr{B} = 2\mathscr{B}_1 + \mathscr{B}_2$ *has* -3 *as its minimal eigenvalue and the associated eigenspace has dimension* $\binom{n}{3} - n$.

Proof. The algebra $\mathscr{A}$ is commutative. Hence it is similar to an algebra of diagonal matrices. Each eigenvalue of B is the appropriate linear combination of the eigenvalues of $\mathscr{B}_0$, $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$. We must verify that $\mathscr{E}_{ij}$ is the jth eigenvalue of $\mathscr{B}_i$. Since $\mathscr{B}_0$ is the identity matrix, the entries in the first row of $\mathscr{E}$ are immediate. Bose and Laskar [3, Lemma 2.1] have computed the eigenvalues of $\mathscr{B}_1$. The fundamental relations

$$\mathscr{B}_1^{\,2} = \sum p_{11}^k \mathscr{B}_k \qquad \text{and} \qquad \mathscr{B}_1\mathscr{B}_2 = \sum p_{12}^k \mathscr{B}_k$$

imply

$$(\mathscr{E}_{1j})^2 = \sum p_{11}^k \mathscr{E}_{kj} \qquad \text{and} \qquad \mathscr{E}_{1j}\mathscr{E}_{2j} = \sum p_{12}^k \mathscr{E}_{kj}\,,$$

from which the remaining entries in $\mathscr{E}$ follow easily.

Let x_j be the dimension of the jth eigenspace of $\mathscr{B}_i$. Then

$$\mathscr{E}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \text{trace } B_0 \\ \text{trace } B_1 \\ \text{trace } B_2 \\ \text{trace } B_3 \end{pmatrix} = \begin{pmatrix} \binom{n}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

gives four independent linear equations relating the x_j's.

2.3. LEMMA. *Suppose C, D are distinct cliques of at least $n - 4$ first associates and $C \cap D$ contains two elements a, b. Then there are at most $12 - n$ elements of $A_1(a) \cap A_1(b)$ not in $C \cup D$. In particular, $n \leqslant 12$.*

Proof. Choose $c \in C$ and $d \in D$ which are not first associates. Then $C \cap D \subseteq A_1(c) \cap A_1(d)$. Let x denote the number of elements of $A_1(a) \cap A_1(b)$ that are not in $C \cup D$. Since $C \cup D \subseteq (A_1(a) \cap A_1(b)) \cup \{a, b\}$, we have

$$\begin{aligned} x + 2(n-4) - 4 &\leqslant x + |C| + |D| - |C \cap D| \\ &= x + |C \cup D| = 2 + P_{11}^1 = n. \end{aligned}$$

Thus $x \leqslant 12 - n$.

2.4. LEMMA. *Suppose $S = \{x, x_1, x_2, y, y_1, y_2\} \subseteq V$; $(x, x_i), (y, y_i) \in A_1$; $(x, y), (x_1, x_2) \in A_2$; and $(x, y_i), (y, x_i) \in A_3$ for $i = 1, 2$. Then S has at most one more pair of second associates. In this case $\mathscr{B} = 2\mathscr{B}_1 + \mathscr{B}_2$ has $z = 2x - 2y - x_1 - x_2 + y_1 + y_2$ as an eigenvector with -3 as the associated eigenvalue.*

Proof. Suppose S contains a third pair of second associates. Let x, y, x_1, x_2, y_1, y_2 index the first six rows and columns of the matrix $\mathscr{B} = 2\mathscr{B}_1 + \mathscr{B}_2$. Then

$$\mathscr{B} = \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 2 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 2 & 2 & \\ 2 & 0 & 0 & 1 & s & u & \\ 2 & 0 & 1 & 0 & t & v & * \\ 0 & 2 & s & t & 0 & w & \\ 0 & 2 & u & v & w & 0 & \\ \hline & & & * & & & * \end{array}\right],$$

where $s, t, u, v \in \{1, 0\}$ and $w \neq 0$ (since $p_{11}^3 = 0$) and where at least one of these equals 1 (by assumption).

Let $\mathscr{P}$ be an orthogonal $\binom{n}{3} \times \binom{n}{3}$ matrix of the form:

$$\mathscr{P} = \operatorname{diag}(1/2^{1/2}, 1/2^{1/2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1, \ldots, 1)$$

$$\times \left[\begin{array}{cc|cccc|c} 1 & 1 & & & 0 & & \\ 1 & -1 & & & & & \\ \hline & & 1 & 1 & -1 & -1 & \\ & 0 & 1 & -1 & -1 & 1 & 0 \\ & & 1 & -1 & 1 & -1 & \\ & & 1 & 1 & 1 & 1 & \\ \hline & & & & 0 & & * \end{array}\right]$$

and let $\mathscr{C}_i$ be the principal submatrix of $\mathscr{P}\mathscr{B}_i\mathscr{P}^{-1}$ determined by the second and third rows and columns. Then

$$\mathscr{C}_1 = \begin{pmatrix} 0 & 2^{1/2} \\ 2^{1/2} & (\mathscr{B}_1)_{5,6}/2 \end{pmatrix} \quad \text{and} \quad \mathscr{C}_2 = \begin{pmatrix} -1 & 0 \\ 0 & d \end{pmatrix},$$

where $d = [1 + (\mathscr{B}_2)_{5,6} - (\mathscr{B}_2)_{3,5} - (\mathscr{B}_2)_{3,6} - (\mathscr{B}_2)_{4,5} - (\mathscr{B}_2)_{4,6}]/2$. By Proposition 2.2 and Theorem 1.1a, $2\mathscr{C}_1 + \mathscr{C}_2$ has minimal eigenvalue greater than or equal to -3. Since the characteristic polynomial of $2\mathscr{C}_1 + \mathscr{C}_2$ is monic of degree 2, this amounts to

$$0 \leqslant \det(2\mathscr{C}_1 + \mathscr{C}_2 - (-3)I) = 2((\mathscr{B}_1)_{5,6} + d - 1).$$

If $(\mathscr{B}_2)_{5,6} = 1$, then $(\mathscr{B}_1)_{5,6} = 0$, so $d = 1$ and $s = t = u = v = 0$. If $(\mathscr{B}_2)_{5,6} = 0$, then one of s, t, u, v equals 1 by assumption, so $d = 0$. In either case, $2\mathscr{C}_1 + \mathscr{C}_2$ has -3 as an eigenvalue and $(2^{1/2}, -1)$ is the associated eigenvector. Therefore,

$$z/2 = 2^{1/2}(x/2^{1/2} - y/2^{1/2}) - (x_1/2 + x_2/2 - y_1/2 - y_2/2)$$

is an eigenvector of $2\mathscr{B}_1 + \mathscr{B}_2$, by Theorem 1.1b.

2.5. Lemma. *Suppose $n > 10$ and S is a subconfiguration as in Lemma* 2.4. *Then none of the remaining pairs of elements (y_1, y_2), (x_i, y_j) is in A_2.*

Proof. Suppose not and take $v \in A_1(x) \cap A_3(y)$ not in S. By Lemma 2.4, $\mathscr{B} = 2\mathscr{B}_1 + \mathscr{B}_2$ has $z = 2x - 2y - x_1 - x_2 + y_1 + y_2$ as an eigenvector and the vth component of $z(2\mathscr{B}_1 + \mathscr{B}_2)$ is zero. Thus

$$0 = 4 + m(v, y_1) + m(v, y_2) - m(v, x_1) - m(v, x_2),$$

where $m(a, b) = 3 - i$ for ith associates (a, b). This can only be satisfied if $y_1, y_2 \in A_3(v)$ and $x_1, x_2 \in A_1(v)$.

This shows that each of the $p_{13}^2 - 2 = n - 7$ choices of $v \in A_1(x) \cap A_3(y)$ not in S is actually in $A_1(x_1) \cap A_1(x_2)$. But $(x_1, x_2) \in A_2$ and $x \in A_1(x_1) \cap A_1(x_2)$, so $n - 7 \leqslant p_{11}^2 - 1 = 3$.

3. Critical Configurations and Their Embedding

Suppose (a, a') and (b, b') are two pairs of complementary 3 sets in a 6 set. Then $S = \{a, a', b, b'\}$ determines a configuration of special interest in the tetrahedral association scheme A^* of characteristic 6 because $a + a' - b - b'$ is an eigenvector of the matrix $\mathscr{B}^* = 2\mathscr{B}_1{}^* + \mathscr{B}_2{}^*$ described in Proposition 2.2. The purpose of this section is to study the ways in which configurations isomorphic to S may be embedded in our pseudotetrahedral association scheme A.

We continue to use the notation of Section 2.

If $S \subseteq V$ and $|S \cap A_i(a)| = 1$ for each $a \in S$ and each $i = 0, 1, 2, 3$, then we say S is a *critical configuration of the first kind.*

3.1. Lemma. *Suppose S is a critical configuration of the first kind having third associates (a, a'), (b, b'). Then $v(S) = a + a' - b - b'$ is an eigenvector of $\mathscr{B} = 2\mathscr{B}_1 + \mathscr{B}_2$ having eigenvalue -3.*

Proof. This is immediate from Theorem 1.1b, Proposition 2.2, and the fact that

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$$

has $(1, -1, -1, 1)$ as an eigenvector with -3 as the associated eigenvalue.

It seems the easiest way to describe the restrictions that Theorem 1.1b implies on the embedding of critical configurations is by means of a multigraph.

Definition. For distinct ith associates $a, b \in V$, let $m(a, b) = 3 - i$.

3.2. Lemma. *Suppose S is a critical configuration of the first kind having third associates (a, a'), (b, b'). Suppose $w \in V - S$. Then $m(a, w) + m(a', w) = m(b, w) + m(b', w)$.*

Proof. By Lemma 3.1, $v(S) = a + a' - b - b'$ is an eigenvector of $B = 2B_1 + B_2$. The claimed equation is equivalent to the assertion that the wth coordinate of $(v(S))B$ is zero.

Suppose S and T are critical configurations of the first kind and

$|S \cap T| = 2$. Then the eigenvectors $v(S)$ and $v(T)$ defined in Lemma 3.1 may be chosen so that $v(S) + v(T)$ has all coordinates zero except those indexed by $S + T = (S \cup T) - (S \cap T)$. If $S + T$ is not a critical configuration of the first kind, then S is isomorphic to a cycle of first associates (Proposition 3.3). We define a configuration isomorphic to $S + T$ to be a *critical configuration of the second kind.*

3.3. PROPOSITION. *Suppose U is a critical configuration of the second kind. Then each $u \in U$ has one second associate and two first associates in U. If (a, d), (b, c) are the second associates in U, then $v(U) = a + d - b - c$ is an eigenvector of $B = 2B_1 + B_2$ with eigenvalue -3 and*

$$m(a, v) + m(d, v) = m(b, v) + m(c, v) \qquad \text{for any } v \in V - U. \tag{3.3.1}$$

Proof. Let $U = S + T$ for S, T critical configurations of the first kind. Let $S \cap T = \{s, t\}$, $U \cap S = \{a, b\}$ and $T \cap U = \{c, d\}$.

Suppose first $(s, t) \in A_3$. Then the remaining elements of $S \cup T$ may be relabeled so that $a, c \in A_1(s) \cap A_2(t)$. By Lemma 3.2, $a, b \in (A_1(c) \cap A_2(d)) \cup (A_2(c) \cap A_1(d))$. But $(a, b) \in A_3$ and $p_{11}^3 = 0$, so $a \in A_1(c) \cap A_2(d)$ if and only if $b \in A_2(c) \cap A_1(d)$. This shows U to be a critical configuration of the first kind, contrary to the hypothesis.

At the cost of relabeling, suppose $a, c \in A_3(s)$, whence $b, d \in A_3(t)$.

If $(s, t) \in A_2$, apply Lemma 3.2 to (b, T) to see that $(b, c) \in A_3$, $(b, d) \in A_1$. It follows that U is of the first kind, contrary to hypothesis.

This shows $(s, t) \in A_1$. Apply Lemma 3.2 to (a, T) and (b, T) to see that either U is of the first kind or $a \in A_1(c) \cap A_2(d)$ and $b \in A_2(c) \cap A_1(d)$. This proves the first assertion.

Suppose $U = \{a, b, c, d\}$ has second associates (a, d), (b, c) and let $v(U) = a + d - b - c$. Then the principal submatrix of $\mathscr{B} = 2\mathscr{B}_1 + \mathscr{B}_2$ having rows and columns indexed by a, b, c, d has the form:

$$\mathscr{C} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}.$$

Since $(1, 1, -1, -1)$ is an eigenvector of $\mathscr{C}$ with eigenvalue -3, Corollary 2.3 and Theorem 1.1b imply $v(U)$ is an eigenvector of B. Just as in Lemma 3.2, (3.3.1) follows from the fact that $(v(U))B$ has wth coordinate zero.

3.4. COROLLARY. *If b and c are second associates in $A_1(a)$, then $\{a, b, c\}$ is in at most one critical configuration of the second kind.*

Proof. Immediate from (3.3.1).

3.5. LEMMA. *Suppose* $\{w, x, y, z\}$ *is a critical configuration of the first kind and* $(w, x) \in A_1$, $(w, y) \in A_3$. *Let*

$$A = A_1(w) \cap A_1(x) \cap A_2(y) \cap A_2(z), \quad B = A_2(w) \cap A_1(x) \cap A_1(y) \cap A_2(z),$$
$$C = A_1(w) \cap A_2(x) \cap A_2(y) \cap A_1(z), \quad D = A_2(w) \cap A_2(x) \cap A_1(y) \cap A_1(z).$$

Then each $b \in B$ *has a unique first associate in* C.

Proof. By Lemma 3.2 each of the sets A, B, C, D is the intersection of any three of the four sets defining it. Since $(w, y) \in A_3$, $B = A_1(x) \cap A_1(y)$ and similarly $C = A_1(w) \cap A_1(z)$. It follows that $|A| = |B| = |C| = |D| = 4$, since $p_{12}^3 = 9$.

Suppose $b \in B$. Then each of the three points in $A_1(b) \cap A_1(w)$ other than x is in $(A_1(w) \cap A_1(x)) \cup (A_1(w) \cap A_2(x))$. Since $b \in A_1(y)$ and $(w, y) \in A_3$, each of these three points is in $A_2(y)$. It follows from Lemma 3.2 that these three points are all in $A \cup C$. A similar argument shows $A_1(b) \cap A_1(z) \subseteq \{w\} \cup C \cup D$ and that $A_1(c) \cap A_1(x) \subseteq \{w\} \cup A \cup B$ and $A_1(c) \cap A_1(y) \subseteq \{z\} \cup B \cup D$ for each $c \in C$.

It suffices to show each $b \in B$ has a first associate in C, since uniqueness follows from Corollary 3.4 applied to $\{w, x, b\}$.

Suppose $b \in B$ has no first associate in C. Then some $c \in C$ has no first associate in B, by the preceding paragraph. Since

$$|A_1(x) \cap A_1(c)| = p_{11}^2 = |A_1(w) \cap A_1(b)|,$$

we have

$$|A_1(b) \cap A| = 3 = |A_1(c) \cap A|, \tag{3.5.1}$$

and similarly

$$|A_1(b) \cap D| = 3 = |A_1(c) \cap D|. \tag{3.5.2}$$

But $|A| = 4 = |D|$, so b and c are second associates and

$$|A_1(b) \cap A \cap A_1(c)| = 2 = |A_1(b) \cap D \cap A_1(c)|.$$

It follows that the second associate $a \in A$ of b is in $A_1(c)$. Apply Corollary 3.4 to $\{a, c, x\}$ to obtain:

$$|A_1(b) \cap A \cap A_1(a)| \geqslant 1. \tag{3.5.3}$$

Now $A_1(a) \cap A_1(y) \subseteq A_2(w)$, so Lemma 3.2 implies $A_1(a) \cap A_1(y) \subseteq B \cup D$. Therefore a has four first associates in $B \cup D - \{b\}$ and at most two of these are in $A_1(b)$, by (3.5.3) and the fact that $x \in A_1(a) \cap A_1(b)$. Now (3.5.2) implies that b has a second associate $b' \in B$. The set $\{x, b, b', y\}$ is a critical configuration of the second kind and (3.3.1) implies $A_1(b') \cap A = \{a\}$. But $A_1(b') \cap A_1(w) \subseteq A \cup C \cup \{x\}$, so this forces b' to have at least two first associates in C, contrary to the uniqueness assertion we have already proven.

We adopt the notation of Lemma 3.5 for the remainder of this section.

3.6. LEMMA. *Each $a \in A$ has two first associates $c \in C$.*

Proof. By Lemma 3.5,

$$8 = \sum_{c \in C} | A_1(c) \cap A | = \sum_{a \in A} | C \cap A_1(a)|, \tag{3.6.1}$$

so it suffices to show $| A_1(a) \cap C | \geqslant 2$ for all $a \in A$.

Suppose $| A_1(a) \cap C | = 0$ for $a \in A$. Then each $c \in C$ is in $A_2(a)$ and $A_1(c) \cap D \subsetneq A_1(c) \cap A_1(a)$ has two elements by Lemma 3.5. Since $w \in A_1(c) \cap A_1(a)$, $| A_1(c) \cap A \cap A_1(a)| \leqslant 1$. But $| A_1(c) \cap A | = 2$ by Lemma 3.5, so a must have a second associate $a' \in A$.

If $c, c' \in C$ were second associates, then $\{w, z, c, c'\}$ and $\{a', z, c, c'\}$ would contradict Corollary 3.4. Thus C is a clique of first associates. Simialrly $\{a, d, d', y\}$ and $\{a, d, d', z\}$ would contradict Corollary 3.4 for second associates $d, d' \in D$, so D also is a clique of first associates.

Pick $c \in C$, $d \in D \cap A_2(c)$. Then $A_1(c) \cap D \subsetneq A_1(c) \cap A_1(d)$ and $A_1(d) \cap C \subsetneq A_1(c) \cap A_1(d)$ by the preceding paragraph. Since $| A_1(c) \cap D | = 2$ by Lemma 3.5, $| A_1(d) \cap C | \leqslant 1$. Thus (3.6.1) gives 8 as a sum of four integers each of which is in $\{0, 1, 4\}$. It follows that $| A_1(d) \cap C | = 0$ for some $d \in D$.

Now interchange the roles of (a, d), (A, D), (w, y), (x, z), etc., and apply the above argument to see d has a second associate $d' \in D$, contrary to the fact that D is a clique of first associates. This contradiction shows $| A_1(a) \cap C | \geqslant 1$ for each $a \in A$.

Suppose $A_1(a) \cap C = \{c\}$ for $a \in A$, $c \in C$. Then c has two second associates in D by Lemma 3.5 and exactly one of these, say d, is in $A_1(a)$ by Corollary 3.4. Apply Corollary 3.4 to $\{a, d, z\}$ to see that $A_1(a) \cap D \subsetneq \{d\} \cup A_1(d)$. Take $v \in A_2(a) \cap (C \cup D)$ and apply (3.3.1) to $\{a, c, d, z\}$ to see $v \in A_1(c)$ if and only if $v \in A_2(d)$. Since the second associate of a in D is also in $A_2(c)$ this implies that d has no second associates in D. Since c has at most one second associate in $C = A_1(w) \cap A_1(z)$ by Corollary 3.4, this also shows there is a unique $c' \in C \cap A_1(d)$ ($C \cap A_1(d) \neq \varnothing$ by the preceding four paragraphs).

Choose $c'' \in C \cap A_2(d)$, $c'' \neq c$. Then apply Corollary 3.4 to $\{w, c', z\}$ to see $(c', c'') \in A_1$. Thus c', z and the elements of $A_1(c'') \cap D$ are the first associates of both d and c''. But $| A_1(d) \cap A | = p_{11}^2 - | A_1(d) \cap C | = 3$ and $| A_1(c'') \cap A | = 2$ by Lemma 3.5, so $| A \cap A_1(d) \cap A_1(c'')| \neq 0$. This contradiction completes the proof that each $a \in A$ has two first associates in C.

3.7. LEMMA. *B is not a clique of first associates.*

Proof. Suppose B is a clique of first associates. Choose $a \in A$, $d \in D \cap A_1(a)$. Apply Corollary 3.4 to $\{x, a, d\}$ to see $A_1(d) \cap B \cap A_1(a) \neq \varnothing$.

If $\{b, b^*\} = A_1(d) \cap B \neq A_1(a) \cap B = \{b, b'\}$, then Eq. (3.3.1) may be applied to $\{x, a, d, b^*\}$ to see that $(b', b^*) \in A_2$, contrary to hypothesis.

Therefore $A_1(d) \cap B = A_1(a) \cap B$. Now Lemma 3.5 and Eq. (3.3.1) imply $A_1(d) \cap C = A_1(a) \cap C$. But $\Delta = (A_1(d) \cap B) \cup (A_1(d) \cap C)$ is a critical configuration of the second kind and we have shown

$$(A_1(a) \cap D) \cup (A_1(d) \cap A) \subseteq A_1(\Delta)$$

since Δ contains a pair of second associates, and this contradicts the fact that $p_{11} = 4$.

3.8. *Remark.* At this point it is very tempting to assert that $V' = A \cup B \cup C \cup D \cup \{w, x, y, z\}$ must be a tetrahedral scheme of characteristic 6. Unfortunately the local information given by (3.3.1) and Lemma 3.2 seems insufficient to establish this conjecture. One can, however, show that there is only one other incidence structure that V' could possibly be. For such a freak, the rows and columns of $\mathscr{B} = 2\mathscr{B}_1 + \mathscr{B}_2$ indexed by V' may be arranged so the associated principal submatrix of B is

$$\mathscr{B}' = \begin{bmatrix}
0 & 2 & 0 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
2 & 0 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
1 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 1 & 1 & 0 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 2 & 1 & 1 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 \\
2 & 2 & 1 & 1 & 2 & 1 & 2 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 2 \\
1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 2 & 1 \\
1 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 2 \\
1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 2 \\
1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 \\
2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 1 & 2 & 1 & 2 & 2 & 1 \\
2 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 2 & 2 & 1 & 1 & 2 & 2 \\
2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 0 & 1 & 2 & 1 & 1 & 2 & 0 & 2 & 2 & 1 & 1 & 2 \\
2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 2 & 1 & 2 \\
1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 2 & 1 \\
1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 0 & 2 \\
1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 0
\end{bmatrix}.$$

We cannot rule this structure out without invoking a hypothesis $n > 10$ and using Lemmas 2.4 and 2.5 to study the embedding of V' in V.

It seems likely that a proper pseudotetrahedral scheme of characteristic 9 or 10 must contain such a freak.

3.9. LEMMA. *Suppose $n > 10$. Then B is a critical configuration of the second kind.*

Proof. Since $B = A_1(x) \cap A_1(y)$, Corollary 3.4 implies B has at most two pairs of second associates. In view of Lemma 3.7 we may suppose $B = \{b_1, b_2, b_3, b_4\}$ and $\{(b_1, b_4)\} = A_2 \cap B \times B$.

Let $b \in B$. Then $A_1(b) \cap A_3(w) \subseteq A_2(x)$ and $A_3(b) \cap A_1(w) \subseteq A_2(x)$. If one of these sets contains a pair (x_1, x_2) of second associates, then Lemmas 2.4 and 2.5 imply that the other is a clique of $n - 5$ first associtaes. Thus at least one of

$$\Delta(b) = A_3(b) \cap A_2(x) \cap A_1(w),$$
$$\Gamma(b) = A_3(w) \cap A_2(x) \cap A_1(b)$$

is a clique of first associates.

Suppose b and b' are first associates in B. We will show that $\Delta(b)$ is a clique if and only if $\Gamma(b')$ is. This will be a contradiction (since $\{b_1, b_2, b_3\}$ is a triangle of first associates in B) and will complete the proof of Lemma 3.9.

Suppose $\Delta(b)$ and $\Delta(b')$ are cliques of first associates for b, b' first associates in B. By Lemma 3.5 there exist $c \in C \cap A_1(b)$ and $c' \in C \cap A_1(b')$. Apply Eq. (3.3.1) to $\{w, x, b, c\}$ to see $\Delta(b) \subseteq A_2(c)$. Since $b \in A_1(c)$, $|A_1(c) \cap A_1(w) \cap A_1(x)| = 2$ and so c has $n - 4 = p_{11}^1 - 2$ first associates in $A_1(w) \cap A_2(x)$. Thus $\Delta(b) = A_2(c) \cap A_1(w) \cap A_2(x)$.

By an analogous argument, replacing (b, c) with (b', c'), we see that (c, c') is a pair of first associates, each of whose second associates in $A_1(w) \cap A_2(x)$ is a clique of first associates.

Suppose these cliques $\Delta(b)$, $\Delta(b')$ share an element c^*. Then c^* has at least two first associates in $A_1(w) \cap A_1(x)$, by Corollary 3.4 applied to $\{x, w, c^*\}$, so $A_1(w) \cap A_1(c^*)$ contains at least two elements not in $\{w\} \cup \Delta(b)$ or in $\{w\} \cup \Delta(b')$. Therefore $\Delta(b) = \Delta(b')$ by Lemma 2.3, contrary to the fact that $c \in \Delta(b')$, but $c \notin \Delta(b)$. This shows $\Delta(b) \cap \Delta(b')$ is empty.

Therefore $A_1(w) \cap A_2(x)$ is the disjoint union of $\{c, c'\}$, $\Delta(b)$ and $\Delta(b')$ and that each element of $A_1(w) \cap A_2(x)$ is a first associate to precisely one of c, c'. Now Lemma 3.5 and Eq. (3.3.1) imply that each element of B is a first associate of exactly one of $\{b, b'\}$, contrary to the supposed structure of B.

Finally suppose $\Gamma(b)$ and $\Gamma(b')$ are cliques of first associates for first associates $b, b' \in B$. Then $\Gamma(b) - \{y\} \subseteq A_3(w) \cap A_2(x) \cap A_1(y)$, so Lemma 3.2 implies $\Gamma(b) - \{y\} \subseteq A_2(z)$. Thus $\{b\} \cup \Gamma(b) - \{y\}$ and $\{b'\} \cup \Gamma(b') - \{y\}$ are two $n - 5$ cliques in $A_2(z) \cap A_1(y)$. These cliques are distinct since b is the only second associate of w in the first and b' is the only second associate of w in the second. If these cliques were not disjoint, we would have a contradiction to Lemma 2.3, just as above.

It follows that $A_1(y) \cap A_2(z) = (\Gamma(b) - \{y\}) \cup (\Gamma(b') - \{y\}) \cup B$. But each of b, b' has two first associates in $A_1(z) \cap A_1(y)$ and $p_{11}^1 - 2 = n - 4$ first associates in $A_1(y) \cap A_2(z)$, of which all but two are in $(\Gamma(b) - \{y\}) \cup$

$(\Gamma(b') - \{y\})$. Therefore each of b, b' has two first associates in B, contrary to the supposed structure of B.

3.10. THEOREM. *Suppose $n > 10$ and V' is as in Remark 3.8, then V' is a tetrahedral scheme of characteristic 6.*

Proof. By Lemmas 3.5, 3.9, and Eq. (3.3.1) each $b \in B$ has a third associate $c \in C$. By Lemma 3.2 c is uniquely determined by b.

In fact each element of V' has a unique third associate in V'. For suppose $a \in A$ then there is $b \in B \cap A_2(a)$ and $\{w, b, y, c\}$, for $c \in A_3(b) \cap C$ is a critical configuration of the first kind. By Lemma 3.2

$$\begin{aligned}\{b, c\} \cup (A_1(c) \cap A_2(b)) \cup (A_2(c) \cap A_1(b))\\ = (A_1(w) \cap A_2(y)) \cup (A_2(w) \cap A_1(y)) \cup \{w, y\} = V'.\end{aligned}$$

Now apply all of the above arguments to $\{w, b, y, c\}$ to see that a has a unique third associate in V'.

By Proposition 3.3 and Lemma 3.2 it suffices to show $\{w\} \cup A \cup B$ has the same incidence pattern of first associates as $A_1{}^*(\{1, 2, 3\})$, namely, that of a lattice graph of characteristic 3. But $|A_1(b) \cap A| = 2$ for $b \in B$, by Lemma 3.5, and $|A_1(a) \cap B| = 2$, by Lemma 3.6, together imply that $\{w\} \cup A \cup B$ is a strongly regular graph. Since there are no "proper pseudo-lattice graphs" of characteristic 3 the theorem is proved.

3.11. COROLLARY. *Suppose $s, t \in V'$ are second associates in $A_1(u)$, $u \in V'$. Then there exist $p, q \in V' \cap A_1(u)$ such that $\{p, q, s, t\}$ is a critical configuration of the second kind.*

4. CLAWLESS CLIQUES

Let $\mathscr{C}$ be the set of all complete cliques having at least $n - 4$ vertices in the graph of A_1 of first associates and suppose $n > 10$. In order to prove Theorem 1.2 by the argument of Bose and Laskar [2, pp. 375–384], it suffices to show:

(4.1a) Any pair of first associates is in a unique element of $\mathscr{C}$.

(4.1b) Each $v \in V$ is in precisely three elements of $\mathscr{C}$.

In fact (4.1b) follows from (4.1a). For supposing (4.1a), the elements of $\mathscr{C}$ containing v partition the $p_{11}^0 = 3(n - 3)$ first associates of v into disjoint sets of cardinality between $n - 5$ and $n - 1 = p_{11}^1 + 1$. If n is greater than 11, $4(n - 5) > 3(n - 3) > 2(n - 1)$ and so v is contained in precisely three elements of $\mathscr{C}$.

In case $n = 11$, let Ω be the subset of V for which (4.1b) holds. Then

$\Omega \neq \varnothing$, since otherwise each element of $\mathscr{C}$ has seven vertices and $|\mathscr{C}| = |V| p_{11}^0/2\binom{7}{2} = 660/7$ is an integer. Also no element of $\mathscr{C}$ has more than nine elements, for if $a, b \in C$ for such a clique $C \in \mathscr{C}$ then the elements of $C - \{a\}$ have at most one first associate in $A_1(x) - C$ and so at most eight elements of $A_1(a) \cap A_2(b)$ have more than one first associate in $A_1(a) \cap A_1(b)$, contrary to Corollary 3.4. It follows that $v \in \Omega$ if and only if some $C \in \mathscr{C}$ on v has more than seven elements if and only if each $C \in \mathscr{C}$ on v has nine elements. But now any first associate of an element of Ω is in Ω and so $\Omega \neq \varnothing$ implies $\Omega = V$. Thus (4.1a) implies (4.1b) when $n = 11$ as well.

4.2. Lemma. *Each pair of first associates is in at most one element of* $\mathscr{C}$.

Proof. By Lemma 2.3 we need only consider the cases $n = 11$ and $n = 12$. Suppose w, x are first associates in distinct $E, F \in \mathscr{C}$. Let $A = A_1(w) \cap A_1(x)$, $B = A_2(w) \cap A_1(x)$, and $C = A_1(w) \cap A_2(x)$. Then

$$|A_1 \cap A \times B| = \sum_{a \in A} |A_1(a) \cap B| = \sum_{a \in A} (n - 3 - |A_1(a) \cap A|). \quad (4.2.1)$$

When $|E \cap F| = n - 8$, $A \subseteq E \cup F$ and each $a \in A \cap E \cap F$ contributes zero to (4.2.1), while each $a \in A - (E \cap F)$ contributes at most 4. Thus $|A_1 \cap A \times B| \leqslant 32$. Since each $b \in B$ has $p_{11}^2 - 1 = 3$ first associates in $A \cup C$ this implies $|A_1 \cap B \times C| \geqslant 3|B| - 32 = 6n - 56$. By Corollary 3.4, this amounts to saying there are at least $6n - 56$ elements $b \in B$ having a first associate $c \in C$. Let b, c be such a pair. Then b has exactly $p_{11}^1 - |A_1(b) \cap A| = n - 4$ first associates in B, so $6n - 57 > n - 4$ implies b has a third associate $b' \in B$ that has a first associate $c' \in C$. The set $\{b, b', c, c'\}$ is a critical configuration of the first kind by (3.3.1). Theorem 3.10 implies (w, x) is a first associate pair in a tetrahedral subconfiguration of characteristic 6. Corollary 3.11 implies there exists a pair p, q of second associates in $A \cap A_1(b)$. By Eq. (3.3.1) $c \in A_1(p) \cap A_1(q)$ and so $A_1(p) \cap A_1(q) \supseteq \{b, c\} \cup (E \cap F)$. This contradiction shows $n = 11$ and $|E \cap F| = 4$.

In this case (4.2.1) implies $|A_1 \cap A \times B| \leqslant 8 + 6 \cdot 4 + 2 \cdot 1 = 34$ and so $\Omega = \{b \in B \mid A_1(b) \cap C \neq \varnothing\}$ has at least eight elements. By Corollary 3.4 each $b \in \Omega$ has at least $(n - 2) - 2 = 7$ first associates in B and so b has at least one second associate $b' \in \Omega$. Just as above there are $c, c' \in C$ such that $\{b, b', c, c'\}$ is a critical configuration of the first kind, and (w, x) is a first associate pair in a tetrahedral subconfiguration of characteristic 6 by Theorem 3.10. Since the second associate pairs (p, q) and (p', q') obtained from applying Corollary 3.11 to $\{w, x, b\}$ and $\{w, c, b'\}$ are disjoint, at least one, say (p, q) has one element in E and one element in F. Thus

$$A_1(e) \cap A_1(f) \supseteq \{b\} \cup (E \cap F),$$

contrary to the fact that $p_{11}^2 = 4$. This completes the proof of Lemma 4.2.

Suppose $(a, b) \in A_1$, but is not in an element of $\mathscr{C}$. Let $C = A_2(a) \cap A_3(b)$, $D = A_3(a) \cap A_2(b)$ and $E = A_3(a) \cap A_3(b)$.

4.3. LEMMA. *Each $c \in C$ is contained in a unique element $K(c)$ of $\mathscr{C}$ such that $K(c) \supseteq A_1(c) \cap A_3(a)$ and each $d \in D$ is contained in a unique element $K(d)$ of $\mathscr{C}$ such that $K(d) \supseteq A_1(d) \cap A_3(a)$.*

Proof. We treat only the first case. Since (a, b) is in no element of $\mathscr{C}$, there exist $x_1, x_2 \in A_1(a) \cap A_3(c)$ that are second associates. By Lemmas 2.4 and 2.5, $A_3(a) \cap A_1(c)$ is a clique of first associates.

4.4. LEMMA. *Each $c \in C$ has a unique first associate $d \in D$.*

Proof. The uniqueness of d follows from Lemma 3.2 and the fact that $\{a, b, c, d\}$ is a critical configuration of the first kind.

Suppose $c \in \Omega := \{x \in C \mid A_1(c) \cap D = \varnothing\}$. Take $k \in K(c) \cap E$ and suppose $k \in K(c') \neq K(c)$ for some $c' \in C$. Then $K(c) \cap K(c') = \{k\}$ by Lemma 4.2. Consequently,

$$| A_1(k) \cap E | \leqslant p_{13}^3 - 2 = 3n - 20,$$

and

$$|(K(c) \cup K(c')) \cap A_1(k) \cap E | \geqslant (n - 6) + (n - 7) = 2n - 13,$$

by choice of $c \in \Omega$. Since

$$| A_1(k) \cap C | = p_{12}^3 - | A_1(k) \cap A_2(a) \cap A_2(b)| = | A_1(k) \cap D |,$$

there are at least two elements $d, d' \in D \cap A_1(k)$. At least one of these, say d, has no first associate in C, by the first paragraph of this proof. Now $K(d) \cap K(c) = \{k\} = | K(d) \cap K(c')|$ by Lemma 4.2 and the fact that $d \notin K(c) \cup K(c')$. Therefore,

$$n - 6 = | A_1(k) \cap K(d) \cap E | \leqslant (3n - 20) - (2n - 13) = n - 7.$$

This contradiction shows that $K(c) \cap K(c') \neq \varnothing$ only if $K(c) = K(c')$, for each $c' \in C$, $c \in \Omega$.

Let $f = |\{K(c) \mid c \in \Omega\}|$, then $f \leqslant | \Omega |$.

Suppose next that $c \in C - \Omega$. Take $x, y \in E \cap K(c)$ and suppose $x, y \in K(c')$ for some $c' \in C$, $c' \neq c$. Then $c' \notin \Omega$, by the preceding paragraph. Let $\{d\} = A_1(c) \cap D$ and $\{d'\} = A_1(c') \cap D$. Then $K(c) = K(c') = K(d) = K(d')$ by Lemma 3.5 and the fact that each of these contains $\{x, y\}$. This implies

$c \in A_1(d) \cap A_1(d')$, contrary to Lemma 3.2. This contradiction shows a pair of first associates in E is in $K(c)$ for at most one $c \in C$. Thus

$$|A_1 \cap E \times E \cap \{(x, y) \mid x, y \in K(c) \text{ for some } c \in C - \Omega\}|$$
$$= \left[\binom{n-4}{2} - |\Omega|\right](n-6)(n-7).$$

We have shown that an object in such a pair must be one of the $[p^1_{33} - f(n-5)]$ not in $K(c)$ for some $c \in \Omega$ and that each such object has at most $3(n-7)$ first associates in E that are in $K(c)$ for some $c \in C - \Omega$. Thus,

$$\left[\binom{n-4}{2} - |\Omega|\right](n-6) \leqslant 3\left[\binom{n-4}{3} - f(n-5)\right].$$

It follows that there exists $c \in \Omega$ such that $|K(c)| \geqslant n-1$. For $x \in K(c)$, $x \neq c$, this shows $A_1(c) \cap A_1(x)$ contains at least $(n-3)(n-4)$ ordered pairs of first associates. Therefore, $|A_1 \cap (A_1(c) \cap A_1(x)) \times (A_1(c) \cap A_2(x))| \leqslant (n-2)(n-3) - (n-3)(n-4) = 2(n-3)$ and so some $y \in A_1(c) \cap A_2(x)$ has more than one first associate in $A_2(c) \cap A_1(x)$, contrary to Corollary 3.4.

4.5. Lemma. *Each pair of second associates $x, y \in A_1(a) \cap A_1(b)$ share a first associate in $A_1(a) \cap A_2(b)$.*

Proof. Suppose $x \in A_1(a) \cap A_2(b)$. Then there are $p^2_{13} = (n-5)$ first associates of x in $A_3(b) \cap A_2(a)$. By Lemma 4.4 each such object is in a critical configuration of the first kind, so Lemma 3.6 implies $|A_1(a) \cap A_1(b) \cap A_1(x)| = 2$. Therefore, $|A_1 \cap (A_1(a) \cap A_1(b) \times A_1(a) \cap A_2(b))| = 2(2(n-4))$ and so $|A_1 \cap (A_1(a) \cap A_1(b)) \times (A_1(a) \cap A_1(b))| = (n-2)(n-3) - 4(n-4)$. This shows $|A_2 \cap (A_1(a) \cap A_1(b)) \times (A_1(a) \cap A_1(b))| = 4(n-4)$.

By Theorem 3.6, $A_1(a) \cap A_1(b) \cap A_1(x)$ consists of a pair of second associates for each $x \in A_1(a) \cap A_2(b)$. Since Corollary 3.4 implies that this function from $A_1(a) \cap A_2(b)$ to $A_2 \cap (A_1(a) \cap A_1(b)) \times (A_1(a) \cap A_1(b))$ is one to two, the fact that $|A_1(a) \cap A_2(b)| = 2(n-4)$ implies it is onto.

4.6. Lemma. *The pair (a, b) is an edge in an element of $\mathscr{C}$ after all.*

Proof. Pick $c \in C$, then there exist $x_1, x_2 \in A_1(a) \cap A_3(c)$ that are second associates, as in the proof of Lemma 4.3. By Lemma 4.5, there exists $d \in A_2(a) \cap A_1(x_1) \cap A_1(x_2)$. But now c violates the incidence condition (3.3.1) with respect to the critical configuration $\{a, b, x_1, x_2\}$. This contradiction proves the lemma and completes the proof of (4.1a).

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